The basics of time-varying ODE with periodic coefficients and application in rotor dynamics

Parameter excited systems

Prof. Jörg Bienert

May 28^{th}, 2026

History

FLOQUET, Gaston. Sur les équations différentielles linéaires à coefficients périodiques. In: Annales scientifiques de l'École normale supérieure. 1883. S. 47-88.

SUR LES

ÉQUATIONS DIFFÉRENTIELLES LINÉAIRES

A COEFFICIENTS PÉRIODIQUES,

PAR M. G. FLOQUET,

PROFESSEUR A LA FACULTÉ DES SCIENCES DE NANCY.

Je considère, dans ce travail, une équation différentielle linéaire homogène

$$P(y) = \frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + p_2 \frac{d^{m-2} y}{dx^{m-2}} + \dots + p_m y = 0,$$

à coefficients uniformes et périodiques, de même période ω, et dont l'intégrale générale est supposée uniforme.

J'étudie la forme analytique des solutions.

Matthieu Diff.Eq. 1 - moving pendulum (textbook example)

19.1 Vorbetrachtung: Pendel mit bewegtem Aufhängepunkt; Stabilität der Mathieu'schen Differentialgleichung

Das Rüttelpendel (Abb. 19.1), das Schulbeispiel aller Mechanik- und Mathematikbücher, genügt für kleine Schwingungen der Bewegungsgleichung

$$m\ddot{\tilde{u}} + d\dot{\tilde{u}} + \left(\frac{mg}{l} + \frac{mh_0\Omega^2}{l}\cos\Omega t\right)\tilde{u} = 0\,, \qquad (19.3)$$

wobei d die im Bild nicht eingezeichnete Dämpfung darstellt. Zur Rückstellung aus dem Pendelglied mg/l tritt die Parametererregung mit $\cos\Omega t$ und der Parameteramplitude $mh_0\Omega^2/l$. Sie verursacht bei gewissen Erregerfrequenzen die Instabilität.

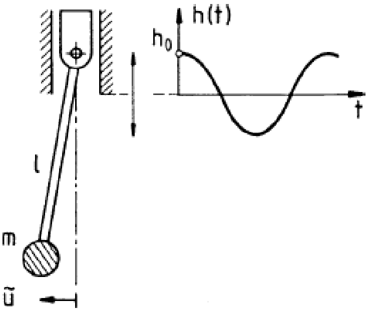
Führt man die Zeitnormierung auf die Erregerfrequenz Ω ein, $\tau = \Omega t$,

$$\dot{\tilde{u}} = \frac{d\tilde{u}}{d\tau}\frac{d\tau}{dt} = \tilde{u}'\Omega\,, \quad \frac{d}{dt} = (\,)'$$

$$\ddot{\tilde{u}} = \frac{d^2\tilde{u}}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 = \tilde{u}''\Omega^2$$

und dividiert noch (19.3) durch $m\Omega^2$, erhält man die übliche Form der *Mathieu'schen* Differentialgleichung

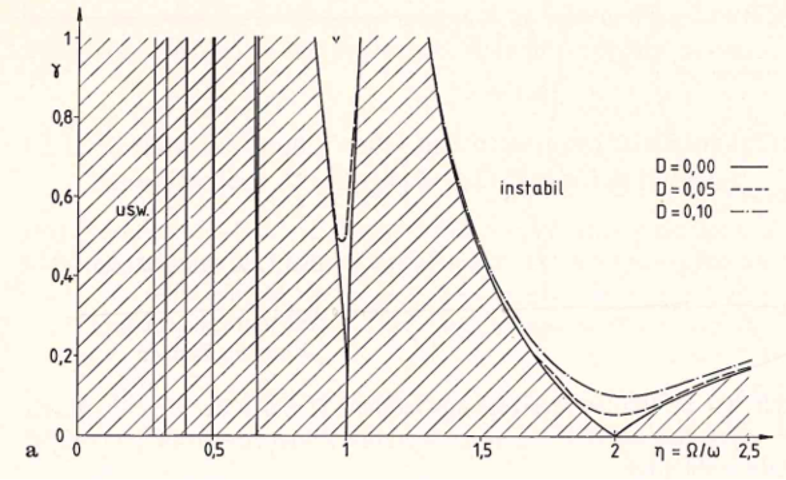
$$\tilde{u}'' + 2D^*\tilde{u}' + (\beta^2 + \gamma\cos\tau)\tilde{u} = 0\,, \qquad (19.4)$$



$$u'' + 2Du' + \left(1 + \eta^2\frac{h_0}{l}\cos(\eta t)\right)u = 0$$

$$\eta = \frac{\Omega_P}{\omega_0}$$

Matthieu Diff.Eq. 2 - stability maps



$$\gamma = h_0/l$$
$$\beta^2 = \Omega^2/\omega_0^2$$

Applications in Rotor Dynamics

- Wind turbines
- Ram air turbines in commercial airplanes
- Propeller in motor glider - today's example

Outline

- History
- From rotor dynamics to periodic ODEs
- General linear solution
- Hill's approach and time variant eigenvectors
- Floquet multiplier
- Birdy glider design and stability analysis
- Outlook: parametric excitation in OMA

Figure: Birdy Glider, Klenhart Design, Spalt(Bavaria)

Rotor as rigid body with rot. DOF for tilt $\Omega_F = \begin{bmatrix} 0 \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}$

$$\vec{M}^{(S)} = \dot{\vec{D}}^{(S)} = \frac{d_F}{dt}(\mathcal{J}^{(S)} \cdot \vec{\omega}) + \vec{\Omega}_F \times (\mathcal{J}^{(S)} \cdot \vec{\omega})$$

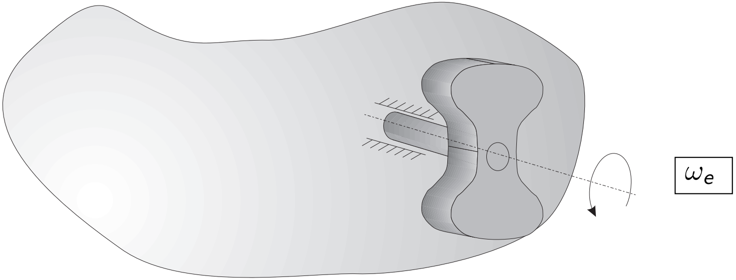


Figure: Structure with Rotor

Mechanical diff.eq. with rotation - non-LTI !

$$(\mathbf{M} + \Delta\mathbf{M}(t))\,\ddot{\mathbf{y}}(t) + (\mathbf{B} + \mathbf{G} + \Delta\mathbf{G}(t))\,\dot{\mathbf{y}}(t) + \mathbf{K}\,\mathbf{y}(t) = \mathbf{f}(t).$$

$$\mathbf{G} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & J_1\omega_e \\ \cdots & 0 & -J_1\omega_e & 0 \end{bmatrix}$$

$$\Delta\mathbf{M}(t) = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & -\Delta J\cos(2\omega_e t) & -\Delta J\sin(2\omega_e t) \\ \cdots & 0 & -\Delta J\sin(2\omega_e t) & \Delta J\cos(2\omega_e t) \end{bmatrix} \text{ and}$$

$$\Delta\mathbf{G}(t) = 2\omega_e \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & \Delta J\sin(2\omega_e t) & -\Delta J\cos(2\omega_e t) \\ \cdots & 0 & -\Delta J\cos(2\omega_e t) & -\Delta J\sin(2\omega_e t) \end{bmatrix}$$

Classical gyro effect $\mathbf{G}$

- first impact vs conventional ODEs

$$\mathbf{M}\,\ddot{\mathbf{y}}(t) + \mathbf{B}\,\dot{\mathbf{y}}(t) + \mathbf{K}\,\mathbf{y}(t) = \mathbf{f}(t).$$

- skew-symmetric and speed dependent matrix

$$\mathbf{G} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & J_1\omega_e \\ \cdots & 0 & -J_1\omega_e & 0 \end{bmatrix}$$

- FRF with left eigenvectors

$$\mathbf{H}(\Omega) = \sum_{k=1}^{2f} \frac{\mathbf{y}_{Rk}\mathbf{y}_{Lk}^T}{a_k(i\Omega - \lambda_k)}$$

Major change to time dependent diff. eq. - $\Delta\mathbf{G}, \Delta\mathbf{M}$

- second impact vs. conventional ODEs:

$$\mathbf{M}\ddot{\mathbf{y}}(t) + (\mathbf{G} + \mathbf{B})\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \mathbf{f}(t)$$

- Time dependency with rotor speed, assumption $\omega_e = \text{const.}$ -> periodic; linear time-periodic (LTP-) System;
  $\Omega_P = 2\omega_e$; unsymmetry $\Delta J = \frac{1}{2}(J_2 - J_3)$

$$\Delta\mathbf{M}(t) = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & -\Delta J\cos(2\omega_e t) & -\Delta J\sin(2\omega_e t) \\ \cdots & 0 & -\Delta J\sin(2\omega_e t) & \Delta J\cos(2\omega_e t) \end{bmatrix}$$

$$\Delta\mathbf{G}(t) = 2\omega_e \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & \Delta J\sin(2\omega_e t) & -\Delta J\cos(2\omega_e t) \\ \cdots & 0 & -\Delta J\cos(2\omega_e t) & -\Delta J\sin(2\omega_e t) \end{bmatrix}$$

First-order diff. eq.

With state vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{bmatrix}$$

system matrices

$$\mathbf{A_1}(t)\,\dot{\mathbf{x}}(t) + \mathbf{A_0}(t)\,\mathbf{x}(t) = \mathbf{p}(t),$$

$$\mathbf{A_1}(t) = \begin{bmatrix} \mathbf{B} + \mathbf{G} + \Delta\mathbf{G}(t) & \mathbf{M} + \Delta\mathbf{M}(t) \\ \mathbf{M} + \Delta\mathbf{M}(t) & \mathbf{0} \end{bmatrix},$$

$$\mathbf{A_0}(t) = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} - \Delta\mathbf{M}(t) \end{bmatrix},$$

and the excitation vector

$$\mathbf{p}(t) = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix}.$$

Fundamental solution for LTIs

Lunze: Control Theory (Regelungstechnik) for constant coefficients for LTIs

Nach Einführung der Abkürzung

$$\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t} \qquad (5.12)$$

kann man diese Lösung in einer häufig in der Literatur angegebenen Form schreiben:

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\,\boldsymbol{x}_0 + \int_0^t \boldsymbol{\Phi}(t-\tau)\,\boldsymbol{b}u(\tau)\,\mathrm{d}\tau.$$

Is there a similar solution for periodic non-LTIs?

Fundamental solution for periodic ODEs

$$\Phi(t) = \mathbf{S}(t)\, \mathrm{e}^{\mathbf{R}t}.$$

Matrix $\mathbf{S}(t)$ is a regular, continuously differentiable, and periodic matrix with

$$\mathbf{S}(t + T_P) = \mathbf{S}(t).$$

It can be used for a coordinate transformation :

$$\mathbf{x}(t) = \mathbf{S}(t)\, \mathbf{z}(t).$$

This is a so-called Lyapunov transformation.

- A linear differential equation with periodic coefficients can be transformed (reduced) into a differential equation with constant! coefficients.

Ljapunov transform

The Ljapunov transform transforms the time-variant ODE to one with constant coefficients.
This will be the key to solve it

$$\dot{\mathbf{z}}(t) + \underbrace{(\mathbf{A_1}(t)\mathbf{S}(t))^{-1}[\mathbf{A_1}(t)\dot{\mathbf{S}}(t) + \mathbf{A_0}(t)\mathbf{S}(t)]}_{-\mathbf{A}_S = \text{const.}}\mathbf{z}(t) = (\mathbf{A_1}(t)\mathbf{S}(t))^{-1}\mathbf{p}(t).$$

- then $\Phi(t) = \mathrm{e}^{\mathbf{A}_S t}$ expected as solution
- challenge to find $\mathbf{S}(t + T_P) = \mathbf{S}(t)$.

Hill's method (1)

Hill's approach for a time variant eigenvector

$$\mathbf{x}(t) = \mathrm{e}^{\rho t}\, \mathbf{u}(t),$$

$$\mathbf{u}(t) = \mathbf{u}(t + T_P) = \sum_{l=-\infty}^{+\infty} \mathbf{u}_l \mathrm{e}^{il\Omega_P t}.$$

- substitute this setup into the differential equation
- specify the periodic system matrices $\mathbf{A_1}, \mathbf{A_0}$ into a complex Fourier series using

$$\mathbf{A_1}(t) = \sum_{a=-\infty}^{+\infty} \mathbf{A}_{1a}\, \mathrm{e}^{ia\Omega_P t}, \qquad \mathbf{A_0}(t) = \sum_{a=-\infty}^{+\infty} \mathbf{A}_{0a}\, \mathrm{e}^{ia\Omega_P t},$$

$$\text{e.g. } \sin(\Omega_P t) = \frac{1}{2i}(\mathrm{e}^{\mathrm{i}\Omega_\mathrm{P}\mathrm{t}} - \mathrm{e}^{-\mathrm{i}\Omega_\mathrm{P}\mathrm{t}}), \cos(\Omega_\mathrm{P}\mathrm{t}) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\Omega_\mathrm{P}\mathrm{t}} + \mathrm{e}^{-\mathrm{i}\Omega_\mathrm{P}\mathrm{t}})$$

Hill's method (2)

- Several summands always occur with the same frequency, a multiple of Ω_P
- A coefficient comparison leads to an algebraic eigenvalue problem of infinite dimension (hyper-eigenvalue problem)

$$(\rho\hat{\mathbf{A}}_1 + \hat{\mathbf{A}}_0)\,\hat{\mathbf{u}} = \mathbf{0},$$

- The hyper-eigenvector $\widehat{\mathbf{u}}$ has the structure

$$\widehat{\mathbf{u}}^T = [\ldots, \mathbf{u}_{-2}^T, \mathbf{u}_{-1}^T, \mathbf{u}_0^T, \mathbf{u}_{+1}^T, \mathbf{u}_{+2}^T, \ldots]$$

Hill's method (3)

- The subvectors $\mathbf{u}_l$ are those from the approach. For practical problems, it is sufficient to restrict the approach vector $\mathbf{u}(t)$ to a few Fourier terms, so that the hypereigenvalue problem is reduced to a finite dimension.
- The terms in the vicinity of the constant part $l = 0$ are of greatest importance here. For a symmetric approach $-L \leq l \leq +L$, the eigenvalue problem has the dimension $(2L + 1)2f$.
- challenge: find the relevant ones due to redundancy

Hill's method (4)

For $L = 2$ the hyper-EVP is:

$$\left(\begin{bmatrix} A_{10} & A_{1-1} & A_{1-2} & A_{1-3} & A_{1-4} \\ A_{1+1} & A_{10} & A_{1-1} & A_{1-2} & A_{1-3} \\ A_{1+2} & A_{1+1} & A_{10} & A_{1-1} & A_{1-2} \\ A_{1+3} & A_{1+2} & A_{1+1} & A_{10} & A_{1-1} \\ A_{1+4} & A_{1+3} & A_{1+2} & A_{1+1} & A_{10} \end{bmatrix} \rho + \right.$$

$$\left. \begin{bmatrix} -2i\Omega_P A_{10} + A_{00} & -i\Omega_P A_{1-1} + A_{0-1} & A_{0-2} & i\Omega_P A_{1-3} + A_{0-3} & 2i\Omega_P A_{1-4} + A_{0-4} \\ -2i\Omega_P A_{1+1} + A_{0+1} & -i\Omega_P A_{10} + A_{00} & A_{0-1} & i\Omega_P A_{1-2} + A_{0-2} & 2i\Omega_P A_{1-3} + A_{0-3} \\ -2i\Omega_P A_{1+2} + A_{0+2} & -i\Omega_P A_{1+1} + A_{0+1} & A_{00} & i\Omega_P A_{1-1} + A_{0-1} & 2i\Omega_P A_{1-2} + A_{0-2} \\ -2i\Omega_P A_{1+3} + A_{0+3} & -i\Omega_P A_{1+2} + A_{0+2} & A_{0+1} & i\Omega_P A_{10} + A_{00} & 2i\Omega_P A_{1-1} + A_{0-1} \\ -2i\Omega_P A_{1+4} + A_{0+4} & -i\Omega_P A_{1+3} + A_{0+3} & A_{0+2} & i\Omega_P A_{1+1} + A_{0+1} & 2i\Omega_P A_{10} + A_{00} \end{bmatrix} \right) \begin{bmatrix} u_{-2} \\ u_{-1} \\ u_0 \\ u_{+1} \\ u_{+2} \end{bmatrix} = \mathbf{0}$$

Application: Birdy glider [1]

Figure: Vibration measurement in operation; airfield Renneritz/Leipzig, Sachsen-Anhalt

[1] Bienert, J., and Regnet, S. (2026). Stability Analysis for an Ultra-Lightweight Glider Airplane with Electric Driven Two-Blade Propeller. Vibration, 9(1), 3. https://doi.org/10.3390/vibration9010003

Antriebswelle/Akku

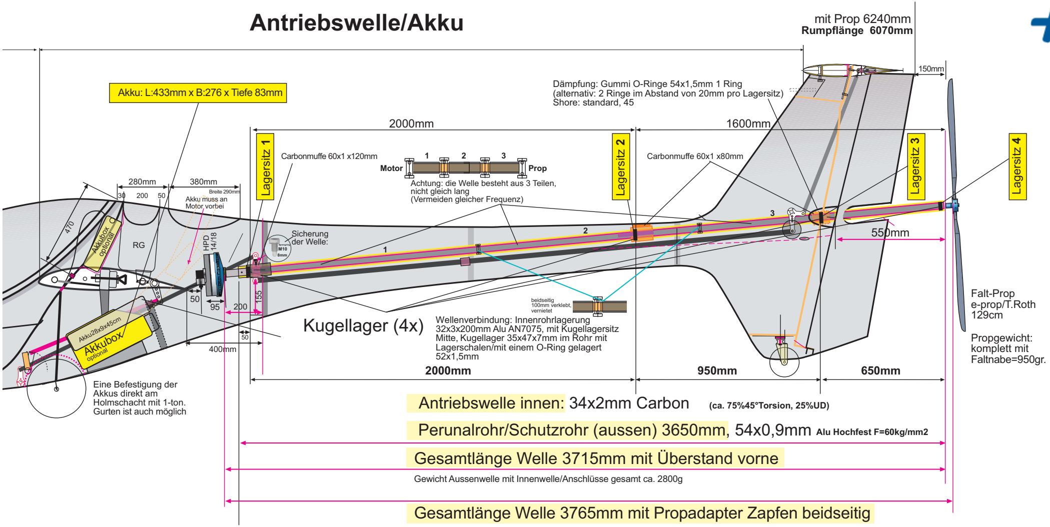
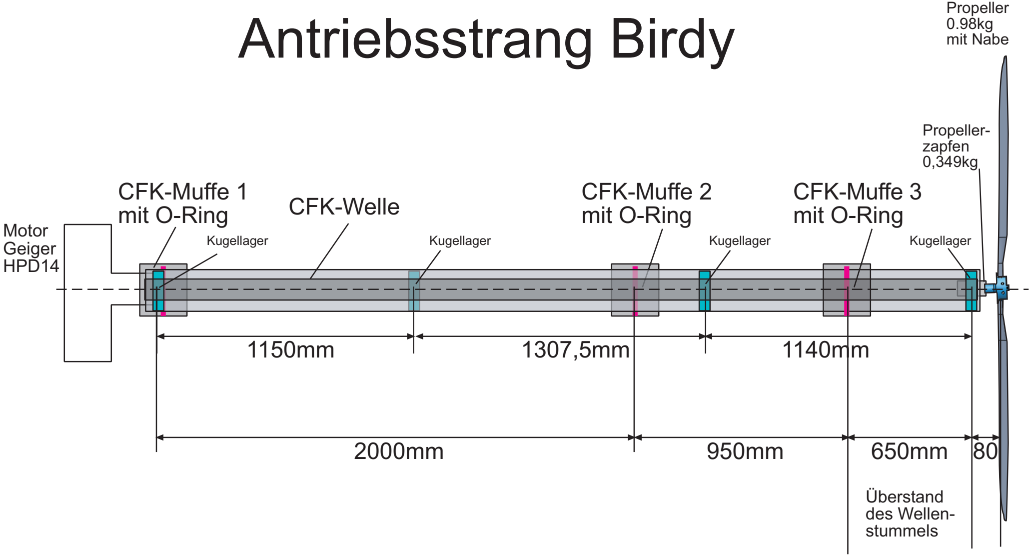


Figure: Propulsion Unit

Antriebsstrang Birdy



The classical matrices for a Matlab MDOF simulation

- stiffness $\mathbf{K}$: Bernoulli Beam elements in 3D
- mass $\mathbf{M}$: Bernoulli Beam elements in 3D
- damping $\mathbf{B}$
- modal damping
  $\text{diag}\lceil \mathbf{2D}\omega_{\mathbf{0}} \rfloor = \mathbf{M}_0^{-1} \cdot \mathbf{B}_0$
- validated model by classical modal analysis with damping of 2%

Mode shape - Birdy propulsion

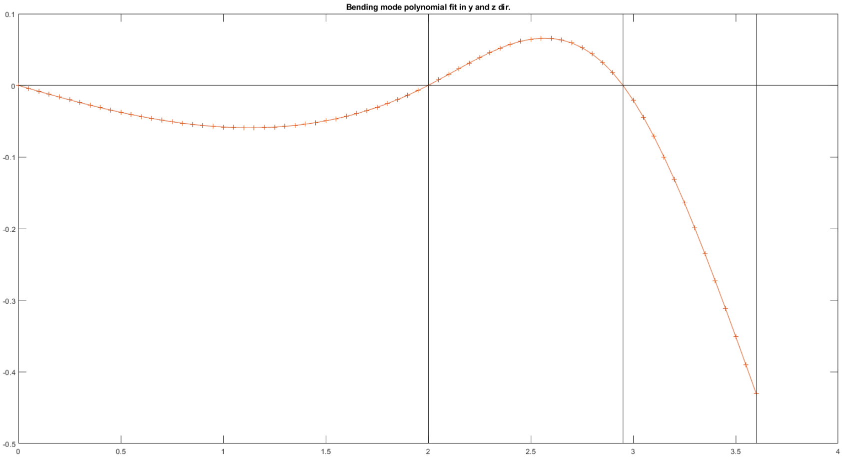


Figure: Polynomial fit to mode shape, 1st bending mode

Increase of degrees of freedom

- x-z plane: 3 Bernoulli beam elements = 4 nodes = 8DOF (4xtransl, 4xrot)
- boundary cond: 3x transl fixed; 5 DOF remaining (1x transl, 4*rot)
- 3D: 2x 5 DOF, spacial symmetry = 10 DOF
- 2nd order ODE into 1st order state space: 2x 10 DOF =20 DOF
- **Hill-Fourier extension:** $-2 \leq l \leq +2$ -> **5 *20 DOF = 100 DOF**

General gyro behaviour

follow the imaginary part of the time-constant system

$$\mathbf{G} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & J_1\omega_e \\ \cdots & 0 & -J_1\omega_e & 0 \end{bmatrix}$$

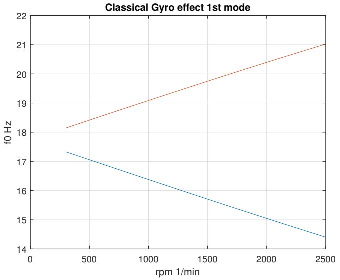


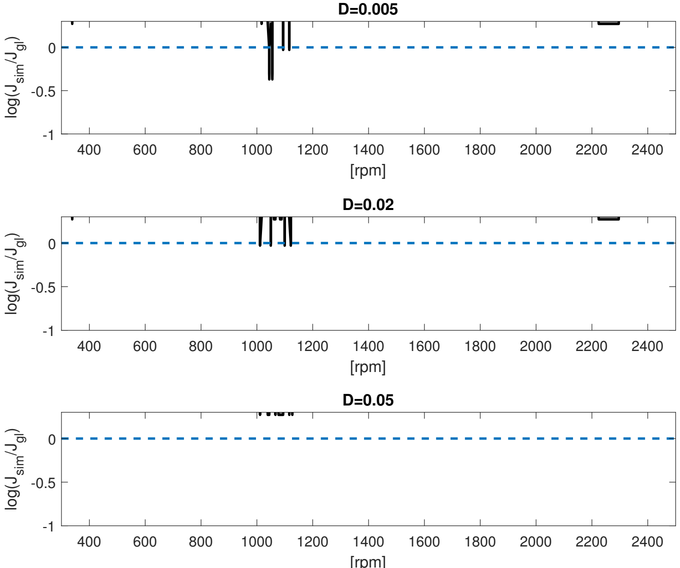
Figure: Frequency of first mode varying with rotor speed (assumption of rigid rotor)

Clarify redundancy of Hill's hyper-EVP

- find the redundant part that has the largest constant constant part $l = 0$
- problem: mixed structure of DOF
  (translation-rotation, displacement-velocity)

$$\widehat{\mathbf{u}}^T = [\ldots, \mathbf{u}_{-2}^T, \mathbf{u}_{-1}^T, \mathbf{u}_0^T, \mathbf{u}_{+1}^T, \mathbf{u}_{+2}^T, \ldots]$$

Main result: real part of EV vs speed and damping (0.5%,2.0%,5.0%)



Linear solution superposition by Floquet (alternative to Hill)

- f mechanical degrees of freedom and $2f$ state variables
- $2f$ linearly independent solutions
- linear combination

$$\mathbf{x}(t) = \Phi(t)\, \mathbf{c}_0$$

- initial conditions $\mathbf{x}_0 = \mathbf{x}(t = t_0)$
- state transition matrix $\Phi_T(t, t_0)$ (as usual for linear systems)

$$\mathbf{x}(t) = \underbrace{\Phi(t)\, \Phi^{-1}(t_0)}_{\Phi_T(t,\, t_0)}\, \mathbf{x_0}.$$

- all solutions can be recombined, regular $\mathbf{C_0}$ required

$$\Psi(t) = \Phi(t)\, \mathbf{C_0}$$

Floquet EVP (1)

$$\Phi(t + T_P) = \Phi(t) \cdot \underbrace{\mathbf{C}}_{monodromie}$$

- Solution after one period: $\mathbf{x}(T_P) = \Phi(T_P)\,\mathbf{x_0}$
- when is it scaled? $\mathbf{x}(T_P) = \mu\,\mathbf{x_0}$
- combine: $\Phi(T_P)\,\mathbf{x_0} = \mu\,\mathbf{x_0}$
- Floquet EVP: $(\Phi(T_P) - \mu\mathbf{I})\,\mathbf{x_0} = \mathbf{0}$ or $[\mathbf{C} - \mu\mathbf{E}]\,\mathbf{c} = \mathbf{0}$

Floquet EVP (2) - decomposition by multipliers

With the help of the eigenvalues μ_k and the eigenvectors $\mathbf{c}_k$, which form the matrix of eigenvectors $\mathbf{C}_c$, a special representation of the monodromy matrix follows:

$$\mathbf{C} = \mathbf{C}_c \text{diag}\left\lceil \mu_k \right\rfloor \mathbf{C}_c^{-1}.$$

The following applies to the stability of natural vibrations: The system is

- unstable for $|\mu_k| > 1$ for at least one eigenvalue,
- marginally stable for $|\mu_k| = 1$ for at least one eigenvalue,
- asymptotically stable for $|\mu_k| < 1$ for all eigenvalues.

A characteristic multiplier

- $\mu_k = 1$ leads to periodic solutions of period T_P, and
- $\mu_k = -1$ leads to periodic solutions of period $2T_P$,

in each case with initial conditions in the form of an eigenvector $\mathbf{c}_k$.

Floquet EVP (3)

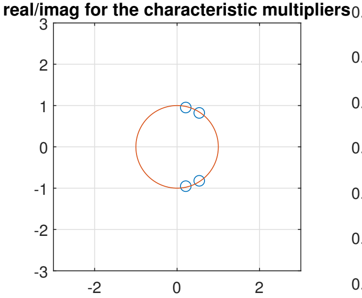
- how to get $2f$ linear independent solutions?
  Solve $\mathbf{x}(T_P) = \Phi(T_P)\,\mathbf{x_0}$ for $2f$ randomized $\mathbf{x_0}$
- problem 1:
  parameter space: $n = 300 \ldots 2500$rpm and $\gamma = \Delta J / J_m$
- problem 2: time-variant eq. with inverse

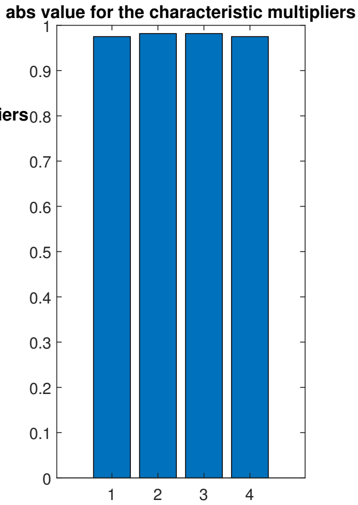
$$\mathbf{A_1}(t)\,\dot{\mathbf{x}}(t) + \mathbf{A_0}(t)\,\mathbf{x}(t) = \mathbf{p}(t),$$

$$\dot{\mathbf{x}}(t) = -\mathbf{A_1}(t)^{-1}\,\mathbf{A_0}(t)\,\mathbf{x}(t)$$

- Solution: modal reduction of differential equation with critical 1st bending mode
- automatic function generation within Matlab
  MM = matlabFunction(odes,'Vars',v,'File','myfile','Optimize' ,false);

Multiplier first bending mode at 2500rpm



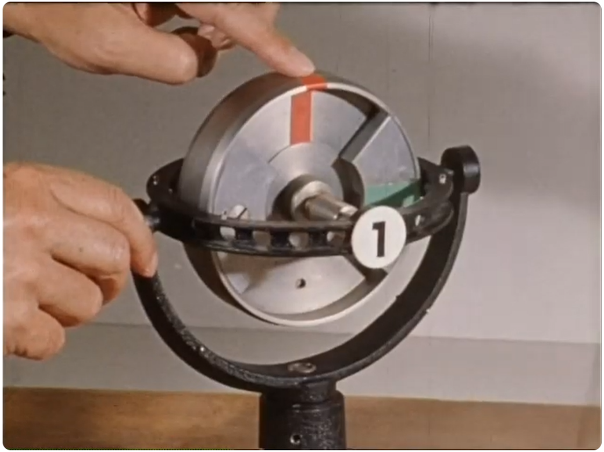


Research Question

What is the expected result for an OMA analysis for periodic time-dependent system ?

Magnus gyro - 1973

Prof. Kurt Magnus, Technical University Munich, * 8. September 1912 in Magdeburg; † 12. Dezember 2003 in München



Gyroscope Stability - 1973

TUM Chair of Applied Mechanics
2050 Abonnenten
Abonnieren
874
Teilen
Speichern

Magnus gyro setup

- Simulation of a small system
- $\omega_e = 2\pi \cdot 3\frac{1}{s} \equiv (180 \text{ rpm})$
- tilt rotation axes fixed elastically

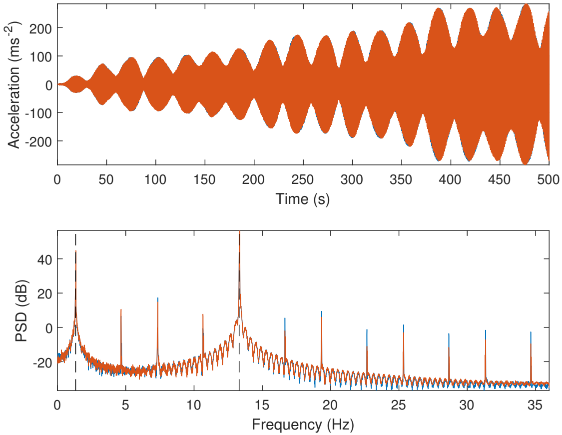
$$\mathbf{K} = \begin{bmatrix} k_\varphi & 0 \\ 0 & k_\varphi \end{bmatrix}$$

- discrete time simulation with time-variant state matrix A[k], 36 time steps per revolution

$$\mathbf{x}[k+1] = \mathbf{A}[k]\mathbf{x}[k] + \mathbf{v}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{w}[k]$$

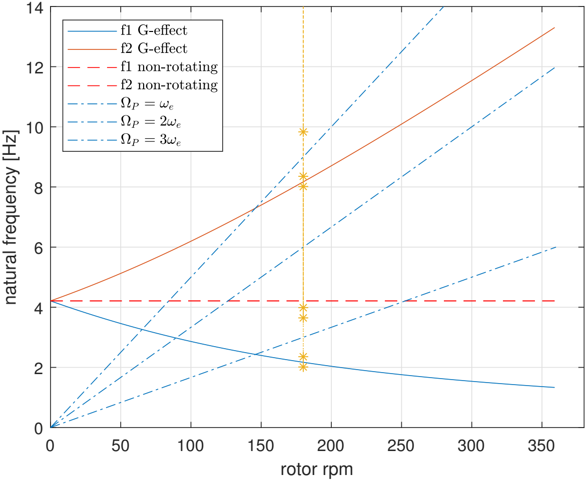
Magnus gyro PSD output [2]



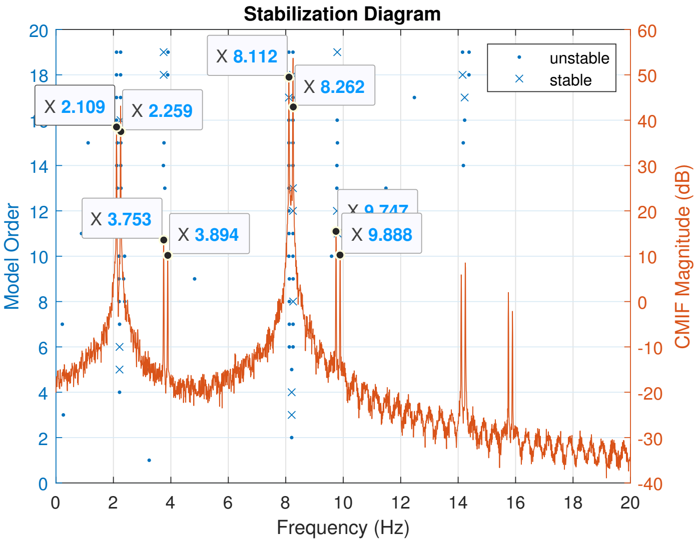
[2] Andrew Otto (2026). OoMA Toolbox (https://de.mathworks.com/matlabcentral/fileexchange/68657-ooma-toolbox), MATLAB Central File Exchange. Abgerufen 9. Mai 2026.

Magnus gyro - frequency chart

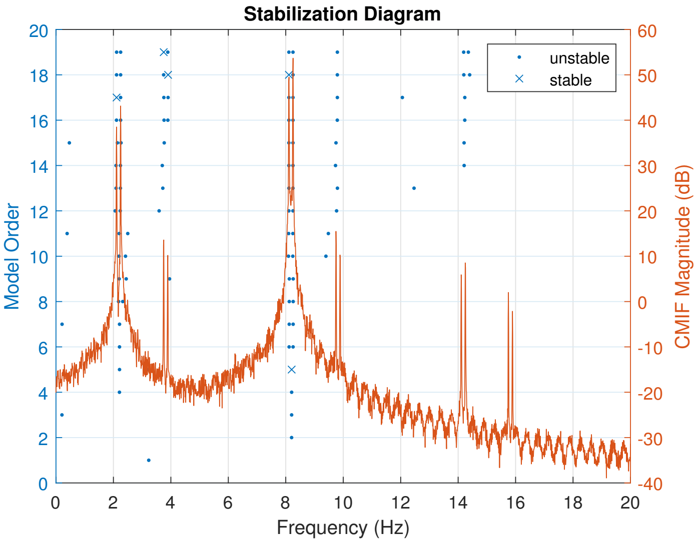
potentially unstable for $n\,\Omega_p = \omega_n \pm \omega_m$, mainly $n\,\Omega_p = 2\cdot\omega_n$



Magnus gyro SSI-cov stabilization diagram at 180rpm -> $f_P = 6Hz$



Magnus gyro SSI-data stabilization diagram at 180rpm -> $f_P = 6Hz$



Research Question

are the peaks with a distance of Ω_P?
are the extracted mode shapes the same as from the Hill EVP?

Conclusion

- excursion into time-variant ODEs
- still a linear system; key message is the existence of periodic eigenvectors
- practical solution by Hill hyper-EVP; answers question of stability
- practical example with glider propulsion unit
- simulation of gyro system shows impact to OMA approaches

Instabilität

1-FHG: $\Omega_p = \frac{2\omega_0}{n}$

m-FHG: $\Omega_p = \frac{\omega_{01} \pm \omega_{02}}{n}$ oder 1. Ordnung

$f_p = \frac{2f_0}{n} = 2f_e$

$f_e = \frac{f_0}{n}$; $f_0 = n \cdot f_1$ kritisch

Zusammenhang Drehzahl für Rotor

$\Omega_p = 2\omega_e$

$f_p = 2f_e$ z.B. $f_p = 6\,Hz$ bei 180 rpm